Exceeding the Courant Condition with the FDTD Method

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***Abstract*— The finite-difference time-domain (FDTD) method is a time-domain implementation of Maxwell's equations that has found a broad range of applications in electromagnetic simulation. A fundamental stability consideration is the Courant condition, which dictates that the time steps used must be long enough to allow the electromagnetic (EM) field to propagate across a cell at the speed of light. A method is described that, under certain circumstances, allows the Courant condition to be exceeded, resulting in a substantially faster computation.**

I. Introduction

THE finite-difference time-domain (FDTD) method [1] has found applications in a broad range of electromagnetic simulation problems [2]. This method simulates the propagation of electromagnetic radiation by time stepping the Maxwell's equations. The problem space is divided up into small cubes (for a three-dimensional (3-D) problem). The physical size of these cubes, or cells, is usually dictated by the need for accuracy. It is generally accepted that the cell size cannot be larger than one-tenth of a wavelength of the highest frequency being simulated. Given that the cell size has been chosen, the size of the time step is limited by the Courant condition [2], which dictates (in three dimensions)

$$\delta t \leq \frac{dx}{\sqrt{3} \cdot c} \tag{1}$$

where δt is the time step, dx is the cell size, and c is the speed of light. This must be chosen on a "worst case" basis, i.e., the value for which c is the highest. If the background medium is free space. then the choice of c in (1) is clear. However, if the background medium is something else, than δt can be selected by

$$\delta t \leq \frac{dx \cdot \sqrt{\varepsilon_{\min}}}{\sqrt{3} \cdot c} \tag{2}$$

where $\varepsilon_{\min}$ is the smallest dielectric in the problem space.

There are many instances where the background medium is other than free space, so (2) would dictate that a much larger time step could be used resulting in a faster FDTD solution. For instance, if the background medium were water with a dielectric constant of 80, this represents a value of dt nine times greater than free space. But, if there is an object within the water that is air or something close to air with a relative dielectric of one, the smaller time step must be used.

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This letter describes a method of circumventing the worst case requirements of (2), at least under some circumstances. To do so, it borrows from the recently developed ideas to simulate dispersive materials with FDTD [3]–[6].

II. The Drude Model

The FDTD method involves the solution of equations of the form

$$\frac{\partial \boldsymbol{D}}{\partial t} = \nabla \times \boldsymbol{H} \tag{3a}$$

$$\boldsymbol{D}(\omega) = \varepsilon_r^*(\omega) \cdot \boldsymbol{E}(\omega) \tag{3b}$$

$$\frac{\partial \boldsymbol{H}}{\partial t} = -\frac{1}{\mu} \nabla \times \boldsymbol{E} \tag{3c}$$

where (3a) and (3c) are the time-dependent Maxwell's curl equations. Equation (3b) is presented in the frequency domain because the complex dielectric constant $\varepsilon_r^*(\omega)$ is almost always given in the frequency domain. This equation must be expressed in the time domain to be compatible with the FDTD paradigm. One of the methods of converting this to a difference equation has been the use of the Z transform [5], [6]

$$\boldsymbol{D}(z) = \varepsilon(z) \cdot \boldsymbol{E}(z) \cdot T. \tag{4}$$

For instance, the permittivity of unmagnetized plasmas is given as

$$\varepsilon^*(\omega) = 1 + \frac{\omega_p^2}{\omega(j\nu_c - \omega)} \tag{5}$$

where

ω_p $2\pi f_p$
f_p plasma frequency
ν_c electron collision frequency.

Taking the Z transform of (5) and inserting it into (4) (details are given in [6])

$$D(z) = E(z) + \frac{\omega_p^2 T}{\nu_c} \cdot \left[\frac{(1 - e^{-\nu_c \cdot T})z^{-1}}{1 - (1 - e^{-\nu_c \cdot T})z^{-1} + e^{-\nu_c \cdot T}z^{-2}}\right] E(z). \tag{6}$$

An auxiliary term will be defined as

$$S(z) = \frac{\omega_p^2 T}{\nu_c} \left[\frac{1 - e^{-\nu_c \cdot T}}{1 - (1 - e^{-\nu_c \cdot T})z^{-1} + e^{-\nu_c \cdot T}z^{-2}}\right] E(z). \tag{7}$$

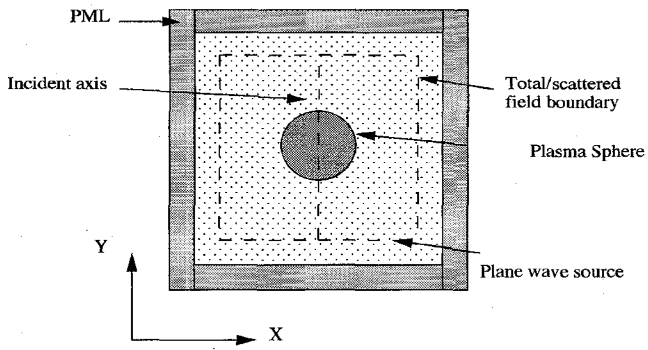

Fig. 1. Diagram of the test to evaluate the FDTD calculation of the E field distribution through and around a sphere of plasma. The plasma frequency is 2000 THz and the electron collision frequency is 57 THz. The sphere is 10 mm in diameter.

$E(z)$ can be solved for by

$$E(z) = D(z) - z^{-1}S(z) \tag{8a}$$

$$S(z) = (1 + e^{-\nu_c \cdot T})z^{-1}S(z) - e^{-\nu_c \cdot T}z^{-2}S(z) + \frac{\omega_p^2 T}{\nu_c}(1 - e^{-\nu_c \cdot T})E(z). \tag{8b}$$

As an example, the problem illustrated in Fig. 1 was simulated using an FDTD program and this formulation. The FDTD program has a problem space divided into two regions, the total field and the scattered field. A plane wave is generated at one end of the total field, propagates the length of the field, and is subtracted out the other end. This plane wave is a Gaussian pulse, and the resulting absorption or scattering is calculated by a running Fourier transform [5].

Fig. 2 is the graph of the E field calculation through the plasma sphere for three different frequencies. The FDTD results are compared to analytic calculations using Bessel function expansions to establish accuracy. There are three very different patterns for the three different frequencies. However, the results make sense in light of Table I, which displays the complex dielectric constant for the three frequencies. At 200 THz, close to the collision frequency, a dielectric of $-98 - i4.5$ results in almost complete reflection, as if the sphere were a metal. In contrast, at 4000 THz, above the plasma frequency, $\varepsilon^*(\omega) = 0.75 - i0.000\,57$ looks very much like free space, and in fact, it appears almost transparent to the incident field. The truly interesting result is at the plasma frequency, 2000 THz, where $\varepsilon^*(\omega) = -i0.004\,54$, i.e.,

$$\varepsilon^*(\omega_p) = 1 + \frac{\omega_p^2}{j\nu_c\omega_p - \omega_p^2} \cong 1 - 1. \tag{9}$$

According to the Courant condition of (2), the FDTD program should have gone unstable for a dielectric constant that is almost zero. What happened, of course, was that the core FDTD program, which simulates (3a) and (3c), was calculating for $\varepsilon^*(\omega) = 1$ and the -1 part was being calculated by (8b). In (5), the second term on the right goes to negative one at the plasma frequency. This idea can be exploited by using a higher dielectric than usual in order to use a larger time step, via (1), and then subtracting the excess dielectric out via the Drude model at the plasma frequency.

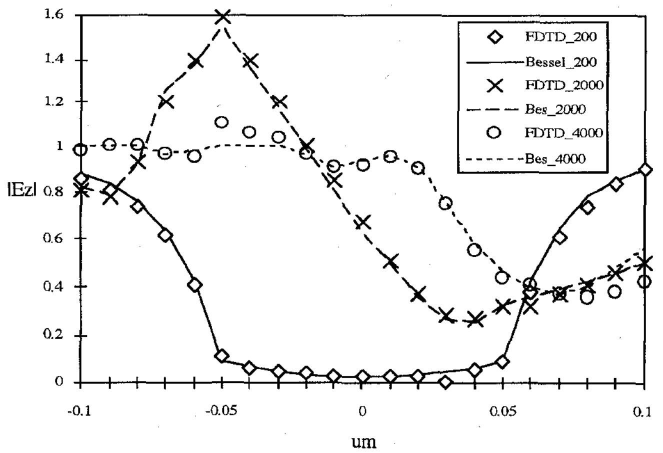

Fig. 2. Comparison of FDTD results versus Bessel function expansion results for an EM wave propagating through a plasma sphere. The parameters used were: plasma frequency = 2000 THz, collision frequency = 57 THz. The diameter or the sphere was 10 μm.

TABLE I
COMPLEX DIELECTRIC CONSTANT OF THE DRUDE MODEL $\varepsilon(\omega) = 1 + (\omega_p^2/j\omega\nu_c - \omega^2)$ USING $\omega_p = 2000$ THz, AND $\nu_c = 57$ THz FOR THREE DIFFERENT FREQUENCIES

Frequency	Dielectric constant	Appears like
200 THz	-98 - i 4.5	Metal
2000 THz	- i 0.000454	Plasma resonace
4000 THz	.75 - i.00057	Free space

III. HYPERTHERMIA EXAMPLE WITH WATER BACKGROUND

Fig. 3 illustrates another simulation problem where the background material is distilled water with $\varepsilon = 80$, and the object is a layered sphere with dielectric properties equal to those of fat, muscle, and air at 100 MHz (Table II). This is typical of a simulation problem for hyperthermia cancer therapy where external RF applicators are coupled to human tissues via distilled water. The human body being modeled would be fat and muscle (bone has about the same dielectric constant as fat) and air from the lung, or pockets within the intestine. The time step, dt, would be calculated by (2) using $\varepsilon_{\min} = 1$, corresponding to air. So if cells of 1 cm were used

$$\delta t \leq \frac{dx \cdot \sqrt{\varepsilon_{\min}}}{\sqrt{3} \cdot c} = 1.92 \times 10^{-11} \text{ s}.$$

This is unfortunate. If the dielectric of the background medium could be used

$$\delta t \leq \frac{(0.01) \cdot \sqrt{80}}{\sqrt{3} \cdot (3 \times 10^8)} = 17.2 \times 10^{-11} \text{ s}$$

or $\sqrt{80} = 8.9$ times as large, meaning the simulation goes almost nine times as fast.

Suppose the air were modeled as

$$\varepsilon_{air}^*(\omega) = 80 + 79\frac{\omega_p^2}{\omega(j\nu_c - \omega)} \tag{10}$$

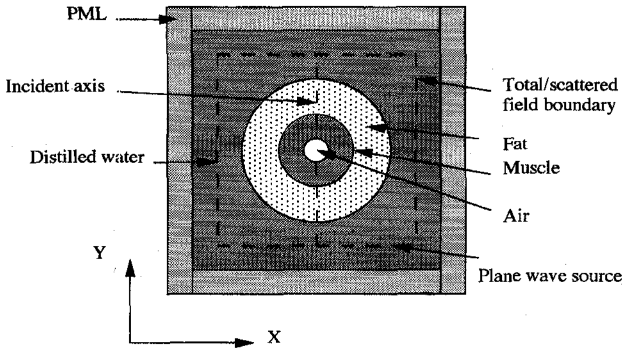

Fig. 3. Diagram of a test to evaluate the FDTD calculation through a sphere with parameters typical of those in a hyperthermia cancer therapy simulation. The outer diameter is 30 cm, the muscle sphere has a diameter of 16 cm, and the air sphere is 4 cm in diameter. The distilled water, which is the background medium, has a dielectric constant of 80. The entire problem space is 60^3 cm^3. The FDTD cells are 1 cm^3.

TABLE II
VALUES OF THE DIELECTRIC CONSTANT AND CONDUCTIVITY FOR VARIOUS HUMAN TISSUES AT 100 MHz

Tissue type	Dielectric constant	Conductivity
Muscle	70	.9
Fat (bone)	5	.05
Air	1.	0

where

$$\omega_p = 100\,\text{MHz}$$
$$\nu_c = 10\,\text{MHz}.$$

At 100 MHz, the term on the right in (10) is

$$79 \frac{\omega_p^2}{j\nu_c \cdot \omega_p - \omega_p^2} \cong -79$$

and (10) becomes

$$\varepsilon_{air}^*(\omega) = 80 - 79 = 1 \tag{11}$$

i.e., it starts out with the higher dielectric constant that was wanted to satisfy the Courant condition, but the Drude formulation will subtract out most of the value to give the correct effective dielectric at the frequency of interest. The FDTD formulation would go as follows: Taking the Z transform of (10) and inserting it into (4)

$$E(z) = D(z) - z^{-1}S(z) \tag{12a}$$
$$S(z) = (1 + e^{-\nu_c \cdot T})z^{-1}S(z) - e^{-\nu_c \cdot T}z^{-2}S(z) + (80 - \varepsilon_r)\frac{\omega_p^2 T}{\nu_c}(1 - e^{-\nu_c \cdot T})E(z). \tag{12b}$$

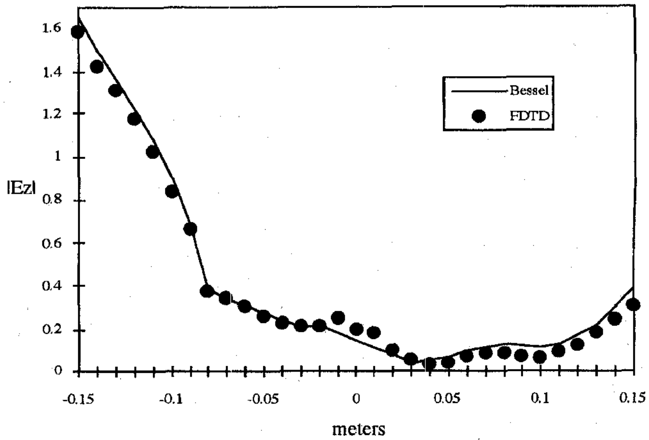

Fig. 4. Comparison of Bessel function and FDTD results for the problem illustrated in Fig. 3, i.e., a layered sphere in distilled water. The incident frequency was 100 MHz.

Equation (12b) is the extra term to implement the speed up. A similar formulation must be used for the fat and muscle tissue. The results of such a simulation are illustrated in Fig. 4. The problem space was 60^3 and the program occupied 10.1 megawords of core memory on a DEC Alpha workstation. The problem converged in 250 time steps. Similar results were obtained using the usual time step dictated by the Courant condition for free space. However, it required 1500 time steps.

IV. DISCUSSION

The technique described in this letter offers the possibility of a substantial speed up in computation time for some problems using the FDTD method.

There are two disadvantages to using this method: 1) information is only obtainable at one frequency per run and 2) the extra calculation to implement the Drude formulation adds extra complexity and requires more core memory (approximately 10%).

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